Interesting occurrence with 7th root of unity

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I've come across a property of $i\sqrt{7}$ that is rather interesting. Letting $\omega = e^{\frac{2\pi i}{7}}$, we let $\eta = \omega + \omega^2 + \omega^4$, we will see that $\eta - \overline{\eta} = i\sqrt{7}$.

Instead of working directly with this difference, we square the expression for something more easily manipulated. $(\eta - \overline{\eta})^2 = \eta^2 - \eta\overline{\eta} + \overline{\eta}^2 = (\omega^2 + 2\omega^3 + \omega^4 + 2\omega^5 + 2\omega^6 + \omega^8) + (\omega^{-2} + 2\omega^{-3} + \omega^{-4} + 2\omega^{-5} + 2\omega^{-6} + \omega^{-8}) - 2(\omega\omega^{-1} + \omega^2\omega^{-1} + \omega^4\omega^{-1} + \omega\omega^{-2} + \omega^2\omega^{-2} + \omega^4\omega^{-2} + \omega\omega^{-4} + \omega^2\omega^{-4} + \omega^4\omega^{-4})$. Using the equalities $\omega^{-1} = \omega^6$, $\omega^{-2} = \omega^5$ etc., we can simplify this down to $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 - 6$. Since the sum of the 7th roots of unity equals 0 and the sum only neglects the root $\omega^7 = 1$, we see that $(\eta - \overline{\eta})^2 = -7$. Thus $\eta - \overline{\eta} = i\sqrt{7}$. Amazing stuff, right?